

Motivating example: Babyboot

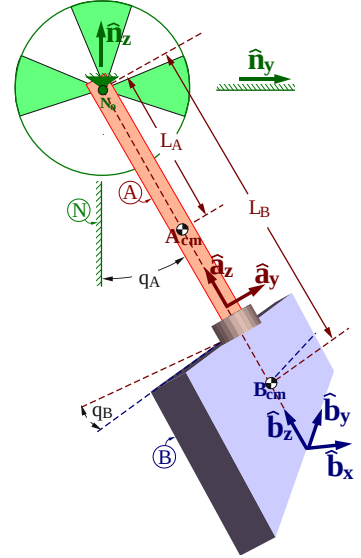
Modeling

The figure to the right is a schematic representation of a swinging babyboot attached by a shoelace to a rigid support. The mechanical model of the babyboot consists of a thin uniform rod A attached to a fixed support N by a revolute joint, and a uniform plate B connected to A with a second revolute joint so that B can rotate freely about A 's axis.

Note: The revolute joints' axes are **perpendicular**, not parallel.

Modeling considerations

- The plate, rod, and support are rigid.
- The revolute joints are frictionless.
- There is no slop or flexibility in the revolute joints.
- Earth is a Newtonian reference frame.
- Air resistance is negligible.
- Forces due to Earth's gravitation are uniform and constant.
- Other distance forces (electromagnetic and gravitational) are negligible.



Identifiers

Right-handed sets of unit vectors $\hat{n}_x, \hat{n}_y, \hat{n}_z$; $\hat{a}_x, \hat{a}_y, \hat{a}_z$; and $\hat{b}_x, \hat{b}_y, \hat{b}_z$ are fixed in N , A , and B , respectively, with $\hat{n}_x = \hat{a}_x$ parallel to the revolute axis joining A to N , $\hat{n}_z$ vertically upward, $\hat{a}_z = \hat{b}_z$ parallel to the rod's long axis (and the revolute axis joining B to A), and $\hat{b}_z$ perpendicular to plate B .

Quantity	Symbol	Type	Value
Earth's gravitational constant	g	Constant	9.81 m/s ²
Distance between N_o and A_{cm}	L_A	Constant	7.5 cm
Distance between N_o and B_{cm}	L_B	Constant	20 cm
Mass of A	m^A	Constant	0.01 kg
Mass of B	m^B	Constant	0.1 kg
A 's moment of inertia about A_{cm} for $\hat{a}_x$	I^A	Constant	0.05 kg*cm ²
B 's moment of inertia about B_{cm} for $\hat{b}_x$	I_x^B	Constant	2.5 kg*cm ²
B 's moment of inertia about B_{cm} for $\hat{b}_y$	I_y^B	Constant	0.5 kg*cm ²
B 's moment of inertia about B_{cm} for $\hat{b}_z$	I_z^B	Constant	2.0 kg*cm ²
Angle from $\hat{n}_z$ to $\hat{a}_z$ with $+\hat{n}_x$ sense	q_A	Dependent variable	varies
Angle from $\hat{n}_z$ to $\hat{a}_z$ with $+\hat{n}_x$ sense	q_B	Dependent variable	varies
Time	t	Independent variable	varies

Physics

Answers at www.MotionGenesis.com $\Rightarrow$ [Get Started](#) $\Rightarrow$ Chaotic Pendulum (Babyboot).

The ODEs (ordinary differential equations) governing the motion of this mechanical system are²

$$\ddot{q}_A = \frac{2\dot{q}_A\dot{q}_B \sin(q_B) \cos(q_B) (I_x^B - I_y^B) - (m^A L_A + m^B L_B) g \sin(q_A)}{I^A + m^A L_A^2 + m^B L_B^2 + I_x^B \cos^2(q_B) + I_y^B \sin^2(q_B)}$$

$$\ddot{q}_B = \frac{-\dot{q}_A^2 \sin(q_B) \cos(q_B) (I_x^B - I_y^B)}{I_z^B}$$

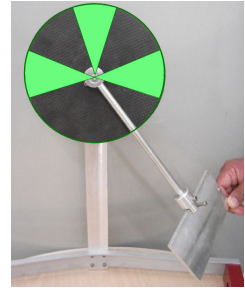
²Four methods for forming equations of motion are: **Free-body diagrams** of A and B (which is inefficient as it introduces up to 10 unknown force/torque measures); D'Alembert's method (**road maps** of Section 20.8) which efficiently forms the two equations shown for $\ddot{q}_A$ and $\ddot{q}_B$ (but require a clever selection of systems, points, and unit vectors); **Lagrange's equations** (an energy-based method that automates D'Alembert's cleverness); **Kane's equations** (a modern efficient blend of D'Alembert and Lagrange).

Simplify and solve

```

Variable qA'', qB''      % Angles and first/second time-derivatives.
%-----
qA'' = 2*( 508.89*sin(qA) - sin(qB)*cos(qB)*qA'*qB' ) / (-21.556 + sin(qB)^2)
qB'' = -sin(qB)*cos(qB)*qA'^2
%-----
Input  tFinal = 10 sec, integStp = 0.02 sec, absError = 1.0E-07
Input  qA = 90 deg,    qB = 1.0 deg,      qA' = 0.0 rad/sec, qB' = 0.0 rad/sec
OutputPlot t sec, qA degrees, qB degrees
%-----
ODE() solveBabybootODE
Quit

```



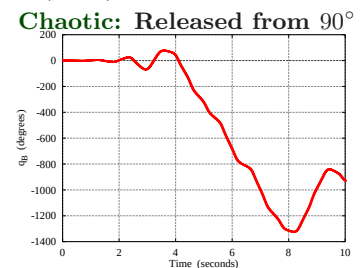
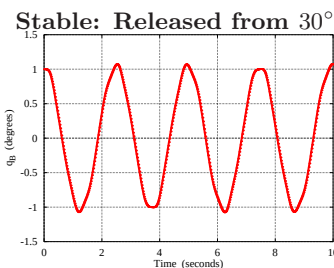
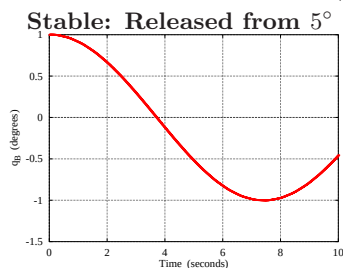
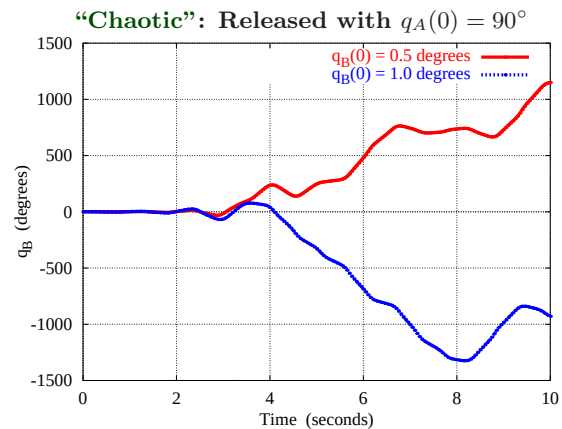
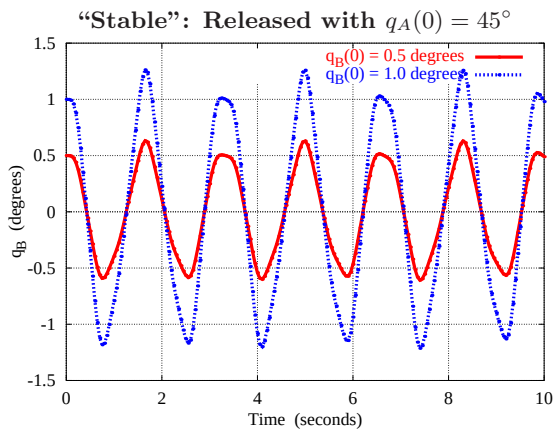
Interpret

The solution to these differential equations reveals this simple system has strange, non-intuitive motion.³ For certain initial values of q_A , the motion of plate B is well-behaved and “stable”. Alternately, for other initial values of q_A , B ’s motion is “**chaotic**” – meaning that a small variation in the initial value of q_B or numerical integration inaccuracies lead to dramatically different results (these ODEs are used to test the accuracy of numerical integrators – the plots below required a numerical integrator error of $\text{absError} = 1 \times 10^{-7}$).

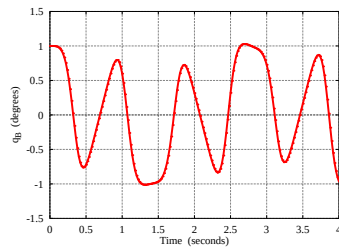
The following chart and figure to the right shows this system’s regions of stability (white) and instability (**green**). Notice the “**chaotic**” plot below shows q_B is *very* sensitive to initial values. A 0.5° change in the initial value of $q_B(0)$ results in more than a 2000° difference in the value of $q_B(t = 10)$!



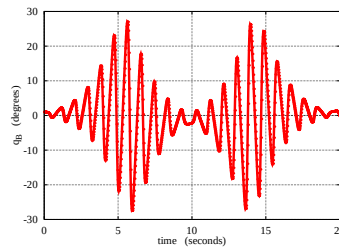
Initial value of q_A			Stability	
$0^\circ \leq q_A(0) \leq 71.3^\circ$			Stable	(white)
$71.4^\circ \leq q_A(0) \leq 111.77^\circ$			Unstable	(green)
$111.78^\circ \leq q_A(0) \leq 159.9^\circ$			Stable	(white)
$160.0^\circ \leq q_A(0) \leq 180.0^\circ$			Unstable	(green)



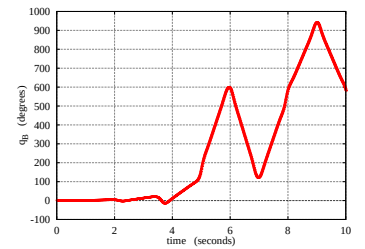
³More information about this problem is in “Mechanical Demonstration of Mathematical Stability and Instability”, *International Journal of Engineering Education (Journal of Mechanical Engineering Education)*, Vol. 2, No. 4, 1974, pp. 45-47, by Thomas R. Kane. Or visit www.MotionGenesis.com $\Rightarrow$ [Get Started](#) $\Rightarrow$ [Chaotic Pendulum \(Babyboot\)](#).



Stable: Released from 135°



Beat: Released from 158°



Chaotic: Released from 177°

Physics

Simulation at www.MotionGenesis.com $\Rightarrow$ [Get Started](#) $\Rightarrow$ Chaotic Pendulum (Babyboot).

```
% File: BabybootWithDAlembertMethod.txt
% Problem: Analysis of 3D chaotic double pendulum
%-----
SetDigits( 5 )      % Number of digits displayed for numbers
%-----
NewtonianFrame N
RigidBody A          % Upper rod
RigidBody B          % Lower plate
%-----
Variable qA''        % Pendulum angle and its time-derivatives
Variable qB''        % Plate angle and its time-derivative
Constant LA = 7.5 cm % Distance from pivot to A's mass center
Constant LB = 20 cm % Distance from pivot to B's mass center
A.SetMass( mA = 10 grams )
B.SetMass( mB = 100 grams )
A.SetInertia( Acm, IAx = 50 g*cm^2, IAY, IAZ )
B.SetInertia( Bcm, IBx = 2500 g*cm^2, IBy = 500 g*cm^2, IBz = 2000 g*cm^2 )
%-----
% Rotational and translational kinematics
A.RotateX( N, qA )
B.RotateZ( A, qB )
Acm.Translate( No, -LA*Az )
Bcm.Translate( No, -LB*Az )
%-----
% Add relevant forces
g> = -9.81*Nz>
System.AddForceGravity( g> )
%-----
% Rotational equations of motion for B and A+B.
Dynamics[1] = Dot( B.GetDynamics(Bcm), Bz> )
Dynamics[2] = Dot( A.GetDynamics(No) + B.GetDynamics(No), Ax> )
Solve( Dynamics, qA'', qB'' )
%-----
% Kinetic and potential energy
KE = System.GetKineticEnergy()
PE = System.GetForceGravityPotentialEnergy( g>, No )
Energy = KE + PE
%-----
% Integration parameters and initial values.
Input tFinal=10, integStp=0.02, absError=1.0E-07, relError=1.0E-07
Input qA =90 deg, qA' = 0.0 rad/sec, qB = 1.0 deg, qB' = 0.0 rad/sec
%-----
% List quantities to be output by ODE command.
OutputPlot t sec, qA deg, qB deg, Energy N*m
%-----
% C/Matlab/Fortran code or immediate numerical solution to ODEs
ODE() BabybootDAlembert
%-----
% Record input together with responses
Save BabybootWithDAlembertMethod.all
Quit
```

